Millersville University **Name** **Answer Key**
Mathematics Department
MATH 211, *Calculus II*, Final Examination
May 4, 2011, 08:00AM-10:00AM

Please answer the following questions. Your answers will be evaluated on their correctness, completeness, and use of mathematical concepts we have covered. Please show all work and write out your work neatly. Answers without supporting work will receive no credit. The point values of the problems are listed in parentheses.

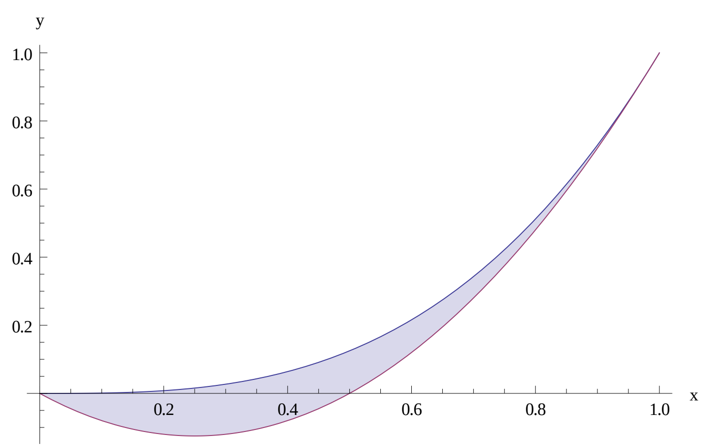
1. (8 points) Find the area bounded between the curves.

$$y = x^3 \quad \text{and} \quad y = 2x^2 - x.$$

The curves intersect when

$$\begin{aligned} 2x^2 - x &= x^3 \\ x^3 - 2x^2 + x &= 0 \\ x(x^2 - 2x + 1) &= 0 \\ x(x-1)^2 &= 0 \\ x &= 0 \quad \text{and} \quad x = 1 \end{aligned}$$

with $x^3 \geq 2x^2 - x$ for $0 \leq x \leq 1$.



$$\begin{aligned} A &= \int_0^1 (x^3 - (2x^2 - x))\,dx \\ &= \int_0^1 (x^3 - 2x^2 + x)\,dx \\ &= \left(\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2\right)\Big|_0^1 \\ &= \frac{1}{4} - \frac{2}{3} + \frac{1}{2} \\ &= \frac{1}{12} \end{aligned}$$

2. (6 points each) Evaluate the following integrals.

(a) $\int 3x \sin(2x)\,dx$

Integrate by parts letting

$$\begin{aligned} u &= 3x & \qquad v &= -\tfrac{1}{2}\cos(2x) \\ du &= 3\,dx & dv &= \sin(2x)\,dx. \end{aligned}$$

Applying the integration by parts formula we obtain

$$\begin{aligned} \int 3x \sin(2x)\,dx &= -\frac{3}{2}x\cos(2x) + \int \frac{3}{2}\cos(2x)\,dx \\ &= -\frac{3}{2}x\cos(2x) + \frac{3}{4}\sin(2x) + C \end{aligned}$$

(b) $\int \cos^4 x \sin^3 x\,dx$

Using the trigonometric identity $\sin^2 x = 1 - \cos^2 x$ we can write

$$\begin{aligned} \int \cos^4 x \sin^3 x\,dx &= \int \cos^4 x \sin^2 x \sin x\,dx \\ &= \int \cos^4 x(1 - \cos^2 x)\sin x\,dx. \end{aligned}$$

Integrate by substitution by letting

$$\begin{aligned} u &= \cos x \\ -du &= \sin x\,dx \end{aligned}$$

yields

$$\begin{aligned} \int \cos^4 x \sin^3 x\,dx &= \int \cos^4 x(1 - \cos^2 x)\sin x\,dx \\ &= \int u^4(1 - u^2)(-1)\,du \\ &= \int (u^6 - u^4)\,du \\ &= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C \\ &= \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C. \end{aligned}$$

(c) $\int_1^\infty xe^{-3x}\,dx$

Note that using integration by parts with

$$\begin{array}{rclcrcl} u & = & x & & v & = & -\frac{1}{3}e^{-3x} \\ du & = & dx & & dv & = & e^{-3x}\,dx, \end{array}$$

we find the indefinite integral

$$\int xe^{-3x}\,dx = -\frac{1}{3}xe^{-3x} + \int \frac{1}{3}e^{-3x}\,dx = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C.$$

Now we may evaluate the improper integral.

$$\begin{aligned} \int_1^\infty xe^{-3x}\,dx &= \lim_{R\to\infty} \int_1^R xe^{-3x}\,dx \\ &= \lim_{R\to\infty} \left(-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right)\Big|_1^R \\ &= \lim_{R\to\infty} \left(-\frac{1}{3}Re^{-3R} - \frac{1}{9}e^{-3R} + \frac{1}{3}e^{-3} + \frac{1}{9}e^{-3}\right) \\ &= \lim_{R\to\infty} \left(-\frac{R}{3e^{3R}} - \frac{1}{9e^{3R}} + \frac{4}{9e^3}\right) \\ &= \frac{4}{9e^3} \approx 0.022128 \end{aligned}$$

3. (6 points) Find the polar form of the equation in rectangular coordinates

$$x^2 + y^2 = x + y.$$

$$\begin{aligned} (r\cos\theta)^2 + (r\sin\theta)^2 &= r\cos\theta + r\sin\theta \\ r^2(\cos^2\theta + \sin^2\theta) &= r(\cos\theta + \sin\theta) \\ r^2 &= r(\cos\theta + \sin\theta) \\ r &= \cos\theta + \sin\theta \end{aligned}$$

4. (8 points) A 550-lb force stretches a spring 3 inches beyond its natural length. How much work is done stretching the spring 6 inches beyond its natural length?

   If the spring obeys Hooke's Law then $F = kx$ where x is the distance the spring is deformed by a force F and k is called the spring constant. In this case

$$550 = 3k \quad \Longrightarrow \quad k = \frac{550}{3}.$$

   Thus the work done is

$$\begin{aligned} W &= \int_0^6 \frac{550}{3} x\,dx \\ &= \frac{275}{3} x^2 \Big|_0^6 \\ &= 3300 \quad \text{inch-lbs.} \end{aligned}$$

5. (8 points) Find the radius and interval of convergence of the series

$$\sum_{k=0}^{\infty} \frac{k}{3^k}(x+2)^k.$$

   Using the Ratio Test we see that

$$\begin{aligned} \lim_{k\to\infty} \left| \frac{\frac{k+1}{3^{k+1}}(x+2)^{k+1}}{\frac{k}{3^k}(x+2)^k} \right| &= \lim_{k\to\infty} \left| \frac{k+1}{k} \frac{3^k}{3^{k+1}} \frac{(x+2)^{k+1}}{(x+2)^k} \right| \\ &= \lim_{k\to\infty} \left| \frac{k+1}{k} \frac{1}{3} (x+2) \right| \\ &= \frac{1}{3}|x+2|. \end{aligned}$$

   Thus the power series converges absolutely when

$$\frac{1}{3}|x+2| < 1 \quad \Longleftrightarrow \quad |x+2| < 3 \quad \Longleftrightarrow \quad -5 < x < 1.$$

   The radius of convergence is $r = 3$. At $x = -5$ the series becomes

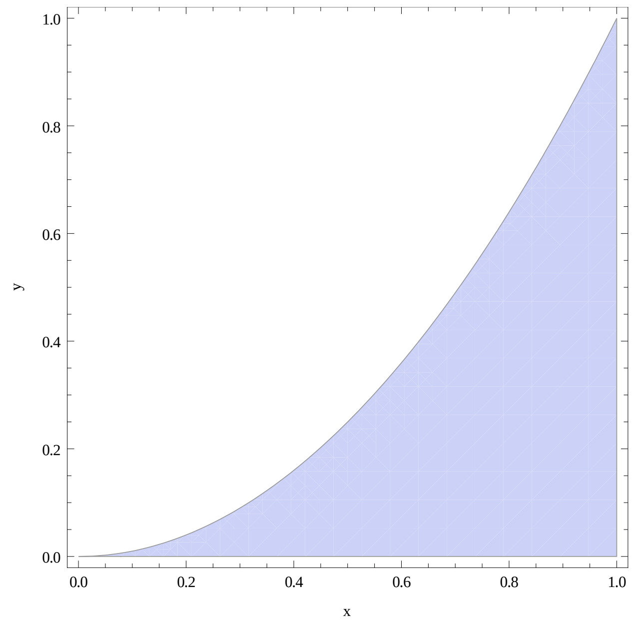
$$\sum_{k=0}^{\infty} \frac{k}{3^k}(-5+2)^k = \sum_{k=0}^{\infty} \frac{k}{3^k}(-3)^k = \sum_{k=0}^{\infty}(-1)^k k$$

   which diverges by the kth term test. At $x = 1$ the series becomes

$$\sum_{k=0}^{\infty} \frac{k}{3^k}(1+2)^k = \sum_{k=0}^{\infty} \frac{k}{3^k}(3)^k = \sum_{k=0}^{\infty} k$$

   which diverges by the kth term test. Thus the interval of convergence is $-5 < x < 1$.

6. (4 points) Set up a definite integral which gives the volume of the solid of revolution generated when the region bounded between the curves $y = x^2$, $y = 0$, and $x = 1$ is revolved around the y-axis. You do not need to evaluate the definite integral.



Using the Method of Shells we may set up the integral

$$V = 2\pi \int_0^1 x(x^2)\,dx = 2\pi \int_0^1 x^3\,dx = \frac{\pi}{2}.$$

7. (6 points each) Determine whether the following series converge absolutely, converge conditionally, or diverge. State the names of all convergence or divergence tests you use.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2+4}{k^3+3k+2}$

This is a positive term series. Suppose $a_k = \dfrac{k^2+4}{k^3+3k+2}$ and $b_k = \frac{1}{k}$. Applying the Limit Comparison Test we see that

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{k^2+4}{k^3+3k+2}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^3+4k}{k^3+3k+2} = 1.$$

Since $\displaystyle\sum_{k=1}^{\infty} b_k$ diverges (Harmonic Series) then the original series also diverges.

(b) $\sum_{k=1}^{\infty}(-1)^k \frac{3^k}{k^2}$

Applying the Ratio Test we see that

$$\lim_{k\to\infty}\left|\frac{(-1)^{k+1}\frac{3^{k+1}}{(k+1)^2}}{(-1)^k\frac{3^k}{k^2}}\right| = \lim_{k\to\infty}\left(\frac{3^{k+1}}{3^k}\,\frac{k^2}{(k+1)^2}\right) = 3\lim_{k\to\infty}\left(\frac{k^2}{(k+1)^2}\right) = 3 > 1$$

so the series diverges.

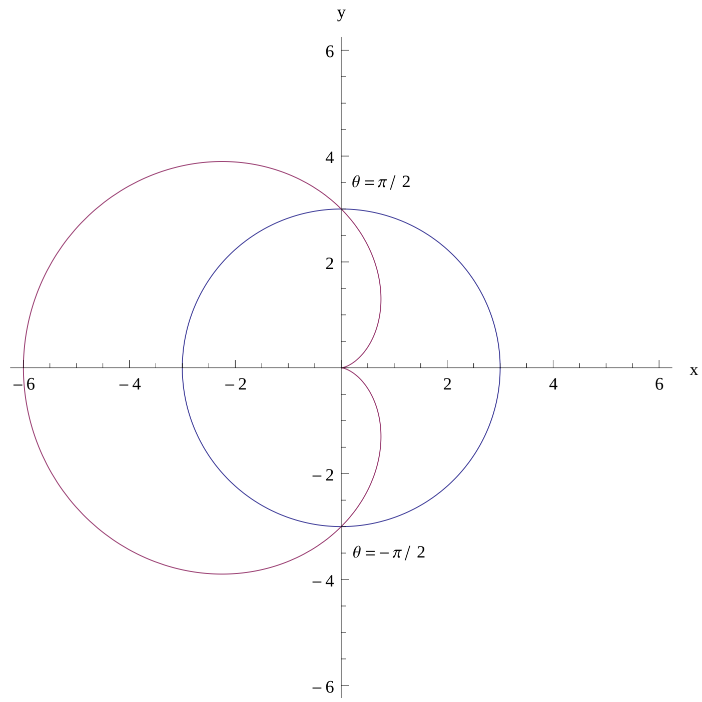
(c) $\sum_{k=1}^{\infty}\frac{1}{k(\ln k+1)^{3/2}}$

This is a positive term series. If we apply the Integral Test we see that

$$\begin{aligned}
\int_1^{\infty}\frac{1}{x(\ln x+1)^{3/2}}\,dx &= \int_1^{\infty} u^{-3/2}\,du \\
&= \lim_{R\to\infty}\int_1^{R} u^{-3/2}\,du \\
&= \lim_{R\to\infty}\left(-\frac{2}{u^{1/2}}\right)_1^R \\
&= \lim_{R\to\infty}\left(2-\frac{2}{R^{1/2}}\right) \\
&= 2 < \infty.
\end{aligned}$$

Since the improper integral converges the infinite series converges absolutely as well.

8. (10 points) Find the area of the region that is inside the circle $r = 3$ and outside the cardioid $r = 3 - 3\cos\theta$. Please include a plot of both curves on the same set of axes. Label the points of intersection of the two curves.



The curves intersect at $\theta = \pm\pi/2$. The area between the curves is

$$\begin{aligned}
A &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left[(3)^2 - (3 - 3\cos\theta)^2\right] d\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left[9 - 9 + 18\cos\theta - 9\cos^2\theta\right] d\theta \\
&= \frac{9}{2}\int_{-\pi/2}^{\pi/2} \left[2\cos\theta - \cos^2\theta\right] d\theta \\
&= \frac{9}{2}\int_{-\pi/2}^{\pi/2} \left[2\cos\theta - \frac{1}{2}(1 + \cos(2\theta))\right] d\theta \\
&= \frac{9}{2}\left(2\sin\theta - \frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right)\Big|_{-\pi/2}^{\pi/2} \\
&= \frac{9}{2}\left(2\sin\frac{\pi}{2} - \frac{\pi/2}{2} - \frac{1}{4}\sin(\pi)\right) - \frac{9}{2}\left(-2\sin\frac{\pi}{2} + \frac{\pi/2}{2} + \frac{1}{4}\sin(\pi)\right) \\
&= \frac{9}{2}\left(2 - \frac{\pi}{4}\right) - \frac{9}{2}\left(-2 + \frac{\pi}{4}\right) \\
&= 18 - \frac{9\pi}{4} \approx 10.9314
\end{aligned}$$

9. (10 points) Find the first four terms of the Taylor series for $f(x) = \sin x$ about $c = \pi/4$.

k	$f^{(k)}(x)$	$f^{(k)}(\pi/4)$	$\dfrac{f^{(k)}(\pi/4)}{k!}$
0	$\sin x$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{1}{\sqrt{2}}$
1	$\cos x$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{1}{\sqrt{2}}$
2	$-\sin x$	$-\dfrac{1}{\sqrt{2}}$	$-\dfrac{1}{2\sqrt{2}}$
3	$-\cos x$	$-\dfrac{1}{\sqrt{2}}$	$-\dfrac{1}{6\sqrt{2}}$

Thus

$$\sin x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3 + \cdots.$$

10. (6 points) Find the partial fraction decomposition of

$$\frac{x^2 - 3x + 4}{x^3 + x}.$$

$$\frac{x^2 - 3x + 4}{x^3 + x} = \frac{x^2 - 3x + 4}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} = \frac{A(x^2 + 1) + (Bx + C)x}{x(x^2 + 1)}$$

Equating numerators we have the equation:

$$x^2 - 3x + 4 = A(x^2 + 1) + (Bx + C)x$$

If we let $x = 0$ then we see that $A = 4$. Substituting this in the equation we have

$$\begin{aligned} x^2 - 3x + 4 &= 4(x^2 + 1) + (Bx + C)x \\ -3x^2 - 3x &= (Bx + C)x \\ -3x - 3 &= Bx + C \end{aligned}$$

and we see that $B = -3$ and $C = -3$. Thus the partial fraction decomposition is

$$\frac{x^2 - 3x + 4}{x^3 + x} = \frac{4}{x} - \frac{3x + 3}{x^2 + 1}.$$

11. (4 points) Set up a definite integral for the arc length of the curve generated by the parametric equations below with $-1 \le t \le 1$.

$$\begin{aligned} x &= t^2 \cos t \\ y &= t^2 \sin t \end{aligned}$$

You do not need to evaluate the definite integral.

$$\begin{aligned} s &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_{-1}^1 \sqrt{(2t\cos t - t^2 \sin t)^2 + (2t \sin t + t^2 \cos t)^2}\, dt \\ &= \int_{-1}^1 \sqrt{4t^2 \cos^2 t - 4t^3 \cos t \sin t + t^4 \sin^2 t + 4t^2 \sin^2 t + 4t^3 \cos t \sin t + t^4 \cos^2 t\, dt} \\ &= \int_{-1}^1 \sqrt{4t^2(\cos^2 t + \sin^2 t) + t^4(\cos^2 t + \sin^2 t)}\, dt \\ &= \int_{-1}^1 \sqrt{4t^2 + t^4}\, dt \\ &= \int_{-1}^1 |t|\sqrt{4 + t^2}\, dt \end{aligned}$$