Millersville University **Name** **Answer Key**
Department of Mathematics
MATH 211, *Calculus II*, Test 1
February 8, 2011

Please answer the following questions. Your answers will be evaluated on their correctness, completeness, and use of mathematical concepts we have covered. Please show all work and write out your work neatly. Answers without supporting work will receive no credit. The point values of the problems are listed in parentheses.

1. (15 points) Evaluate the following indefinite integral.

$$\int 3x\,\cos 2x\,dx$$

Integrate by parts using

$$\begin{aligned} u &= 3x & v &= \tfrac{1}{2}\sin 2x \\ du &= 3\,dx & dv &= \cos 2x\,dx. \end{aligned}$$

Then

$$\begin{aligned} \int 3x\,\cos 2x\,dx &= \frac{3}{2}x\,\sin 2x - \int \frac{3}{2}\sin 2x\,dx \\ &= \frac{3}{2}x\,\sin 2x + \frac{3}{4}\cos 2x + C \end{aligned}$$

2. (15 points) Evaluate the following indefinite integral.

$$\int \cos^3 x \, dx$$

Begin by factoring the integrand.

$$\begin{aligned}
\int \cos^3 x \, dx &= \int \cos^2 x \, \cos x \, dx \\
&= \int (1 - \sin^2 x) \cos x \, dx
\end{aligned}$$

Integrate by substitution using

$$\begin{aligned}
u &= \sin x \\
du &= \cos x \, dx
\end{aligned}$$

then

$$\begin{aligned}
\int \cos^3 x \, dx &= \int (1 - \sin^2 x) \cos x \, dx \\
&= \int (1 - u^2) \, du \\
&= u - \frac{1}{3} u^3 + C \\
&= \sin x - \frac{1}{3} \sin^3 x + C.
\end{aligned}$$

3. (15 points) Evaluate the following definite integral. Give an exact value, no decimal approximations.

$$\int_1^2 x \ln x \, dx$$

Integrate by parts using

$$\begin{aligned} u &= \ln x & v &= \tfrac{1}{2}x^2 \\ du &= \tfrac{1}{x}\,dx & dv &= x\,dx. \end{aligned}$$

Then

$$\begin{aligned} \int_1^2 x \ln x \, dx &= \frac{1}{2}x^2 \ln x \Big|_1^2 - \int_1^2 \frac{1}{2}x^2 \left(\frac{1}{x}\right) \, dx \\ &= 2\ln 2 - \frac{1}{2}\ln 1 - \frac{1}{2}\int_1^2 x\,dx \\ &= 2\ln 2 - \frac{1}{4}x^2 \Big|_1^2 \\ &= 2\ln 2 - 1 + \frac{1}{4} \\ &= 2\ln 2 - \frac{3}{4} \approx 0.636294. \end{aligned}$$

4. (15 points) Evaluate the following indefinite integral.

$$\int \cot^2 x \, \csc^2 x \, dx$$

Integrate by substitution letting

$$\begin{aligned} u &= \cot x \\ -du &= \csc^2 x \, dx \end{aligned}$$

then

$$\begin{aligned} \int \cot^2 x \, \csc^2 x \, dx &= -\int u^2 \, du \\ &= -\frac{1}{3}u^3 + C \\ &= -\frac{1}{3}\cot^3 x + C. \end{aligned}$$

5. (15 points) Evaluate the following indefinite integral.

$$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$$

Use a trigonometric substitution

$$\begin{aligned} x &= 5\sin\theta \\ dx &= 5\cos\theta\,d\theta. \end{aligned}$$

Then

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{25-x^2}}\,dx &= \int \frac{1}{25\sin^2\theta\sqrt{25-25\sin^2\theta}}\,5\cos\theta\,d\theta \\
&= \frac{1}{25}\int \frac{\cos\theta}{\sin^2\theta\sqrt{1-\sin^2\theta}}\,d\theta \\
&= \frac{1}{25}\int \frac{\cos\theta}{\sin^2\theta\,\cos\theta}\,d\theta \\
&= \frac{1}{25}\int \frac{1}{\sin^2\theta}\,d\theta \\
&= \frac{1}{25}\int \csc^2\theta\,d\theta \\
&= -\frac{1}{25}\cot\theta + C \\
&= -\frac{1}{25}\frac{\sqrt{25-x^2}}{x} + C
\end{aligned}$$

6. (15 points) Evaluate the following definite integral. Give an exact value, no decimal approximations.

$$\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx$$

Use a trigonometric substitution

$$\begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta\,d\theta. \end{aligned}$$

Then

$$\begin{aligned} \int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx &= \int_0^{\pi/6} \frac{2\sin\theta}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\,d\theta \\ &= 2\int_0^{\pi/6} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\,\cos\theta\,d\theta \\ &= 2\int_0^{\pi/6} \frac{\sin\theta}{\cos\theta}\,\cos\theta\,d\theta \\ &= 2\int_0^{\pi/6} \sin\theta\,d\theta \\ &= -2\cos\theta\big|_0^{\pi/6} \\ &= -2\cos\frac{\pi}{6} + 2\cos 0 \\ &= -2\left(\frac{\sqrt{3}}{2}\right) + 2 \\ &= 2-\sqrt{3} \approx 0.267949 \end{aligned}$$

Alternatively, you may use ordinary integration by substitution with $u = 4 - x^2$ and $-\frac{1}{2}\,du = x\,dx$.

$$\begin{aligned} \int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx &= \int_4^3 \frac{-1/2}{u^{1/2}}\,du \\ &= \frac{1}{2}\int_3^4 u^{-1/2}\,du \\ &= u^{1/2}\big|_3^4 \\ &= 2-\sqrt{3} \end{aligned}$$

7. (10 points) Find the partial fraction decomposition of the following rational function.

$$\frac{2x-3}{5x^2-4x-1}$$

Begin by factoring the denominator.

$$\begin{aligned}
\frac{2x-3}{5x^2-4x-1} &= \frac{2x-3}{(5x+1)(x-1)} \\
&= \frac{A}{5x+1} + \frac{B}{x-1} \\
2x-3 &= A(x-1) + B(5x+1)
\end{aligned}$$

If $x = -1/5$ then

$$-\frac{2}{5} - 3 = -\frac{17}{5} = A\left(-\frac{1}{5} - 1\right) = -\frac{6}{5}A \quad \Longrightarrow \quad A = \frac{17}{6}.$$

If $x = 1$ then

$$2 - 3 = -1 = B\,(5+1) = 6B \quad \Longrightarrow \quad B = -\frac{1}{6}.$$

Hence

$$\frac{2x-3}{5x^2-4x-1} = \frac{17/6}{5x+1} + \frac{-1/6}{x-1}.$$