Millersville University **Name** **Answer Key**
Department of Mathematics
MATH 211, *Calculus II*, Test 2
March 4, 2011

Please answer the following questions. Your answers will be evaluated on their correctness, completeness, and use of mathematical concepts we have covered. Please show all work and write out your work neatly. Answers without supporting work will receive no credit. The point values of the problems are listed in parentheses.

1. (12 points) Find the exact amount of area bounded between the graphs of $y = x^2$, $y = 0$, and $x = 3$.

$$\begin{aligned} A &= \int_0^3 x^2\,dx \\ &= \left.\frac{1}{3}x^3\right|_0^3 \\ &= 9 \end{aligned}$$

2. (12 points) Set up the definite integral for the arc length of the portion of the graph of $y = x^2 + x - 1$ on the interval $[-1, 2]$. Use your calculator to approximate the arc length.

If $f(x) = x^2 + x - 1$, then

$$\begin{aligned} s &= \int_a^b \sqrt{1 + (f'(x))^2}\, dx \\ &= \int_{-1}^2 \sqrt{1 + (2x + 1)^2}\, dx \\ &\approx 7.52578. \end{aligned}$$

3. (12 points) A force of 45 pounds stretches a spring 9 inches. Find the exact amount of work done to stretch the spring 6 inches beyond its natural length.

   For a Hooke's Law spring $F = kx$ and thus

$$45 = k(9) \quad \Longrightarrow \quad k = 5.$$

   Therefore the work done is

$$W = \int_0^6 5x\,dx = \frac{5}{2}x^2\Big|_0^6 = 90 \quad \text{in-lbs.}$$

4. (5 points each) Suppose a distributed object lies along the x-axis and has a density described by the function, $\rho(x) = x^2 - 2x + 9$ for $0 \leq x \leq 4$.

(a) Find the mass of the object.

$$\begin{aligned} m &= \int_0^4 (x^2 - 2x + 9)\,dx \\ &= \left(\frac{1}{3}x^3 - x^2 + 9x\right)\Big|_0^4 \\ &= \frac{64}{3} - 16 + 36 \\ &= \frac{124}{3} \end{aligned}$$

(b) Find the moment about the origin of the object.

$$\begin{aligned} M_0 &= \int_0^4 x(x^2 - 2x + 9)\,dx \\ &= \int_0^4 (x^3 - 2x^2 + 9x)\,dx \\ &= \left(\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{9}{2}x^2\right)\Big|_0^4 \\ &= 64 - \frac{128}{3} + 72 \\ &= \frac{280}{3} \end{aligned}$$

(c) Find the center of mass of the object.

$$\overline{x} = \frac{M_0}{m} = \frac{280/3}{124/3} = \frac{280}{124} = \frac{70}{31}$$

5. (12 points) Determine whether the following improper integral converges or diverges. If the integral converges, find its value.

$$\int_0^4 \frac{2x}{x^2-1}\,dx$$

The integral is improper due to an infinite discontinuity at $x = 1$.

$$\begin{aligned}
\int_0^4 \frac{2x}{x^2-1}\,dx &= \int_0^1 \frac{2x}{x^2-1}\,dx + \int_1^4 \frac{2x}{x^2-1}\,dx \\
&= \lim_{R\to 1^-} \int_0^R \frac{2x}{x^2-1}\,dx + \int_1^4 \frac{2x}{x^2-1}\,dx \\
&= \lim_{R\to 1^-} \left(\ln|x^2-1|\Big|_0^R \right) + \int_1^4 \frac{2x}{x^2-1}\,dx \\
&= \lim_{R\to 1^-} \Big(\ln|R^2-1| - \ln|-1| \Big) + \int_1^4 \frac{2x}{x^2-1}\,dx \\
&= \lim_{R\to 1^-} \Big(\ln|R^2-1| \Big) + \int_1^4 \frac{2x}{x^2-1}\,dx
\end{aligned}$$

Since the limit diverges to $-\infty$ the improper integral diverges.

6. (13 points) Set up the definite integral for the surface area of the solid of revolution created when the graph of $y = x^3$ on the interval $[0, 2]$ is revolved around the x-axis. Use your calculator to approximate the surface area.

$$\begin{aligned} S &= 2\pi \int_a^b |f(x)|\sqrt{1 + (f'(x))^2}\, dx \\ &= 2\pi \int_0^2 x^3 \sqrt{1 + (3x^2)^2}\, dx \\ &= 2\pi \int_0^2 x^3 \sqrt{1 + 9x^4}\, dx \\ &\approx 203.044 \end{aligned}$$

7. (6 points each) Consider the region bounded between the graphs of $y = x$, $y = 2x$, and $x = 2$.

   (a) Set up a single definite integral to find the volume of the solid of revolution created if the region is revolved around the x-axis. You do not need to evaluate the definite integral.

   Using the Method of Disks

$$V = \pi \int_0^2 [(2x)^2 - x^2]\, dx = \pi \int_0^2 3x^2\, dx.$$

   (b) Set up a single definite integral to find the volume of the solid of revolution created if the region is revolved around the y-axis. You do not need to evaluate the definite integral.

   Using the Method of Shells

$$V = 2\pi \int_0^2 x(2x - x)\, dx = 2\pi \int_0^x x^2\, dx.$$

8. (12 points) Determine whether the following improper integral converges or diverges. If the integral converges, find its value.

$$\int_0^\infty xe^{-x}\,dx$$

The integral is improper because the upper limit of integration is ∞.

$$\int_0^\infty xe^{-x}\,dx = \lim_{R\to\infty}\int_0^R xe^{-x}\,dx$$

We can use integration by parts to evaluate the definite integral.

$$\begin{aligned} u &= x & \qquad v &= -e^{-x} \\ du &= dx & \qquad dv &= e^{-x}\,dx \end{aligned}$$

Thus

$$\int_0^R xe^{-x}\,dx = \left(-xe^{-x}\right)\Big|_0^R + \int_0^R e^{-x}\,dx = -Re^{-R} - e^{-R} + 1$$

and therefore

$$\begin{aligned} \int_0^\infty xe^{-x}\,dx &= \lim_{R\to\infty}\int_0^R xe^{-x}\,dx \\ &= \lim_{R\to\infty}\left(-Re^{-R} - e^{-R} + 1\right) \\ &= 1 - \lim_{R\to\infty}\frac{R}{e^R} \quad \text{(indeterminate } \infty/\infty\text{)} \\ &= 1 - \lim_{R\to\infty}\frac{1}{e^R} \\ &= 1 \end{aligned}$$