Millersville University **Name** **Answer Key**
Department of Mathematics
MATH 211, *Calculus II*, Test 3
April 8, 2011

Please answer the following questions. Your answers will be evaluated on their correctness, completeness, and use of mathematical concepts we have covered. Please show all work and write out your work neatly. Answers without supporting work will receive no credit. The point values of the problems are listed in parentheses.

1. (10 points each) Determine whether the following infinite series converge absolutely, converge conditionally, or diverge. To receive full credit, you must justify your answers by citing the tests for convergence or divergence you are using.

   (a) $\sum_{k=1}^{\infty} 2k^{-9/8}$

   $$\sum_{k=1}^{\infty} 2k^{-9/8} = \sum_{k=1}^{\infty} \frac{2}{k^{9/8}} = 2\sum_{k=1}^{\infty} \frac{1}{k^{9/8}}$$

   which is a p-series with $p = 9/8 > 1$. This infinite series **converges absolutely**.

   (b) $\sum_{k=1}^{\infty} \frac{\sin k}{2^k}$

   Consider the absolute value series,

   $$\sum_{k=1}^{\infty} \left| \frac{\sin k}{2^k} \right|.$$

   If we let $a_k = \left| \frac{\sin k}{2^k} \right|$ then $a_k \leq b_k = \frac{1}{2^k}$. Since

   $$\sum_{k=1}^{\infty} \frac{1}{2^k}$$

   is a convergent geometric series, the original series **converges absolutely** by the Comparison Test.

(c) $\sum_{k=1}^{\infty} \frac{e^{1/k}}{k^2}$

Let $f(x) = \frac{e^{1/x}}{x^2}$ then $f(k) = \frac{e^{1/k}}{k^2}$ for $k = 1, 2, \ldots$. Consider the improper integral

$$\begin{aligned} \int_1^{\infty} f(x)\,dx &= \lim_{R\to\infty} \int_1^R \frac{e^{1/x}}{x^2}\,dx \\ &= \lim_{R\to\infty} \left(-e^{1/x}\right)\Big|_1^R \\ &= \lim_{R\to\infty} \left(e - e^{1/R}\right) \\ &= e < \infty. \end{aligned}$$

Since the improper integral converges, then according to the Integral Test, the infinite series converges. Since the infinite series is a positive term series, it **converges absolutely**.

(d) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{1 + \ln k}$

This is an alternating series with $a_k = \frac{2}{1 + \ln k} > 0$ for $k = 1, 2, \ldots$.

- Since $\ln(k+1) > \ln k$ for $k = 1, 2, \ldots$ then $a_{k+1} < a_k$ for $k = 1, 2, \ldots$.
- $\lim_{k\to\infty} \frac{2}{1 + \ln k} = 0$.

By the Alternating Series Test, the infinite series converges.
If we consider the absolutely value series

$$\sum_{k=1}^{\infty} \left| (-1)^{k+1} \frac{2}{1 + \ln k} \right| = \sum_{k=1}^{\infty} \frac{2}{1 + \ln k}$$

and let $b_k = \frac{2}{1 + k}$ then $0 < b_k < a_k$ for $k = 1, 2, \ldots$. The infinite series

$$\sum_{k=0}^{\infty} \frac{2}{1 + k}$$

diverges (by the Integral Test) and thus the original series **converges conditionally**.

2. (10 points) Find the sum of the infinite series

$$\sum_{k=0}^{\infty}(-1)^k \frac{5}{7^k}.$$

This is a geometric series with $a = 5$ and $r = -1/7$.

$$\sum_{k=0}^{\infty}(-1)^k \frac{5}{7^k} = \sum_{k=0}^{\infty} 5\left(\frac{-1}{7}\right)^k = \frac{5}{1-(-1/7)} = \frac{5}{8/7} = \frac{35}{8}$$

3. (10 points) Find the interval of convergence of the power series

$$\sum_{k=0}^{\infty} \frac{k}{3^k}(x+2)^k.$$

Using the Ratio Test

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{\frac{k+1}{3^{k+1}}(x+2)^{k+1}}{\frac{k}{3^k}(x+2)^k}\right| &= \lim_{k\to\infty}\left|\frac{k+1}{3^{k+1}}\frac{3^k}{k}\frac{(x+2)^{k+1}}{(x+2)^k}\right| \\
&= \lim_{k\to\infty}\frac{1}{3}\frac{k+1}{k}|x+2| \\
&= \frac{1}{3}|x+2|.
\end{aligned}$$

Thus the power series converges absolutely for $|x+2| < 3$ or equivalently $-5 < x < 1$.

If $x = -5$ then

$$\sum_{k=0}^{\infty} \frac{k}{3^k}(-5+2)^k = \sum_{k=0}^{\infty} \frac{k}{3^k}(-3)^k = \sum_{k=0}^{\infty}(-1)^k k$$

which diverges by the kth term test.

If $x = 1$ then

$$\sum_{k=0}^{\infty} \frac{k}{3^k}(1+2)^k = \sum_{k=0}^{\infty} \frac{k}{3^k}(3)^k = \sum_{k=0}^{\infty} k$$

which diverges by the kth term test.

Thus the interval of convergence is $-5 < x < 1$.

4. (10 points) Express the following function as a convergent power series and state the interval of convergence.

$$\frac{3}{1+4x}$$

Thinking of the function as the sum of a geometric series, we have

$$\frac{3}{1+4x} = \frac{3}{1-(-4x)} = \sum_{k=0}^{\infty} 3(-4x)^k = \sum_{k=0}^{\infty} 3(-4)^k x^k$$

which converges absolutely for $|-4x| < 1$ or equivalently $-\frac{1}{4} < x < \frac{1}{4}$.

5. (10 points) Find the fourth Taylor polynomial in x for the function $f(x) = e^{-x}$.

Let $c = 0$ then

k	$f^{(k)}(x)$	$f^{(k)}(0)$	$\frac{f^{(k)}(0)}{k!}$
0	e^{-x}	1	1
1	$-e^{-x}$	-1	-1
2	e^{-x}	1	$\frac{1}{2}$
3	$-e^{-x}$	-1	$-\frac{1}{6}$
4	e^{-x}	1	$\frac{1}{24}$

and thus

$$P_4(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4.$$

6. (10 points) Use a well-known Taylor series to express the Taylor series for

$$f(x) = \sin(3x^2).$$

Since $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ then

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (3x^2)^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k 3^{2k+1}}{(2k+1)!} x^{4k+2}.$$

7. (2 points each) Answer "**converges**", "**diverges**", or "**cannot tell**" for the following situations.

(a) If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \frac{1}{3}$, then $\sum_{k=1}^{\infty} a_k$ **converges**.

(b) If $\lim_{k\to\infty} a_k = \frac{1}{3}$, then $\sum_{k=1}^{\infty} a_k$ **diverges**.

(c) If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 1$, then $\sum_{k=1}^{\infty} a_k$ **cannot tell**.

(d) If $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ **cannot tell**.

(e) If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 0$, then $\sum_{k=1}^{\infty} a_k$ **converges**.