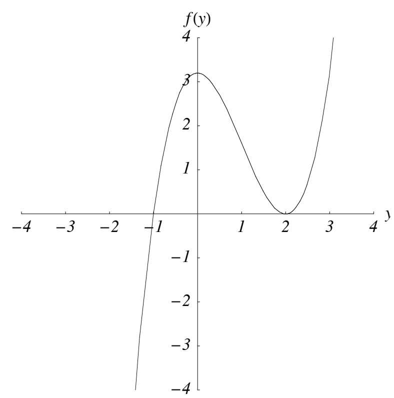
Math 1b. Calculus II
Final Exam Solutions

Summer 2007

1. (12 points) Suppose that we know the graph below is the graph of the right-hand side of the differential equation $dy/dx = f(y)$.



(a) Draw the phase line for the differential equation $dy/dx = f(y)$.

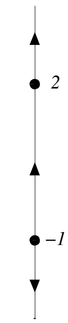
(b) Classify each equilibrium solution of the differential equation $dy/dx = f(y)$ as a source, a sink, or a node.

(c) Find a possible function for $f(y)$.

(d) Draw a rough sketch of the slope field that corresponds to the differential equation $dy/dx = f(y)$.

Solution.

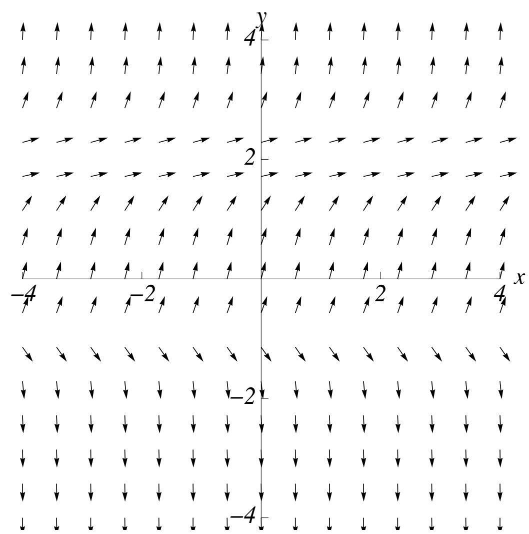
(a)



(b) $y=-1$ is a source. $y=2$ is a node.

(c) $f(y)=(y+1)(y-2)^2$

(d)



2. (8 points) A dosage d of a drug is given daily at $t=0,1,2,3,\ldots$ days. The drug decays exponentially at a rate r in the blood stream. Thus, the amount in the bloodstream after $n+1$ doses is $d+de^{-r}+de^{-2r}+\cdots+de^{-nr}$

   (a) Find the level of the drug after an "infinite" number of doses. That is, find $d+de^{-r}+de^{-2r}+\cdots+de^{-nr}+\cdots$

   (b) If $r=0.1$, what dosage is needed to maintain a drug level of 2?

Solution.

(a) $d+de^{-r}+de^{-2r}+\cdots+de^{-nr}+\cdots=\dfrac{d}{1-e^{-r}}$

(b) If $2=\dfrac{d}{1-e^{-0.1}}$, then $d=2-2e^{-0.1}$.

3. (9 points) Evaluate the following integrals.

(a) $\displaystyle\int \frac{5x+7}{(x+1)(x+2)}\,dx.$

(b) $\displaystyle\int_0^1 x\arctan x\,dx$

(c) $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\,dx$

Solution.

(a) Using partial fractions, we can determine that $\displaystyle\int \frac{5x+7}{(x+1)(x+2)}\,dx = 2\ln|x+1|+3\ln|x+2|+C.$

(b) Using integration by parts, we can determine that $\displaystyle\int_0^1 x\arctan x\,dx = \frac{\pi-2}{4}.$

(c) Using integration by substitution, we can determine that $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \frac{1}{\ln(2)}.$

4. (12 points) Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x=-2$ is given by

$$T_3(x) = 2 - \frac{3}{8}(x+2)^2 - \frac{1}{12}(x+2)^3.$$

(a) Find $f(-2)$, $f'(-2)$, and $f''(-2)$.

(b) Determine whether f has a local minimum, a local maximum, or neither at $x=-2$. Justify your answer.

(c) Use $T_3(x)$ to find an approximation for $f(0)$.

(d) The fourth derivative of f satisfies the inequality

$$\left|f^{(4)}(x)\right| \le \frac{1}{4}$$

for all x in the closed interval $[-2,0]$. Find an error bound on the approximation for $f(0)$ that you found in part (c).

Solution.

(a) $f(-2)=2$, $f'(-2)=0$, and $f''(-2)=-3/4$ (since $f''(-2)/2! = -3/8$).

(b) Since $f'(-2)=0$, and $f''(-2)=-3/4$, f has a local a local maximum at $x=-2$ by the Second Derivative Test.

(c) $T_3(0) = -1/6$

(d) $|f(0)-T_3(0)| \le \dfrac{M}{4!}|0+2|^4 = \dfrac{(1/4)}{24}\cdot 2^4 = \dfrac{1}{6}$

5. (8 points) A bag of sand originally weighing 144 lb is lifted at a constant rate. As it rises, sand leaks out at a constant rate. The sand is half gone by the time the bag has been lifted 18 ft.

(a) How many pounds of sand leak out of the bag ***per foot*** as the bag is lifted?

(b) How much work was done in lifting the bag 18 ft?

Solution.

(a) 4 lb/ft

(b) $\displaystyle\int_0^{18} 144-4x\,dx = 1944$ ft-lb.

6. (8 points) Match each slope field with one of the following differential equations.

(a) $\dfrac{dy}{dt} = ty$

(b) $\dfrac{dy}{dt} = \cos^2 y$

(c) $\dfrac{dy}{dt} = y(y^2 - 1)$

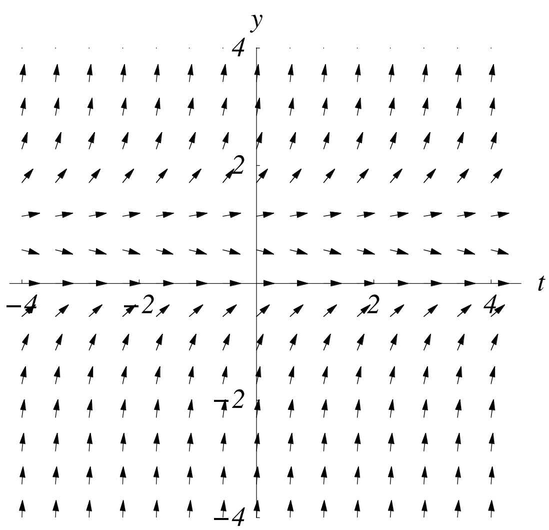
(d) $\dfrac{dy}{dt} = \cos^2 t$

(e) $\dfrac{dy}{dt} = y - t$

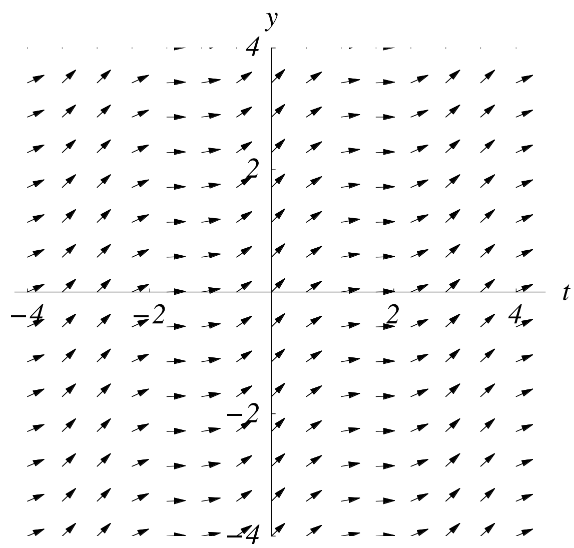
(f) $\dfrac{dy}{dt} = 1 - y$

(g) $\dfrac{dy}{dt} = ty^2$

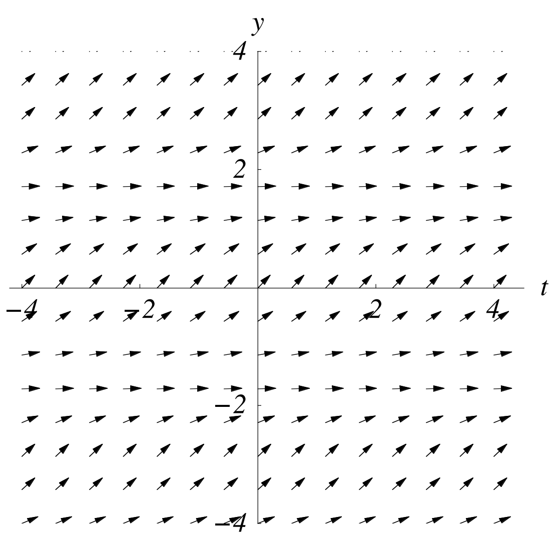
(h) $\dfrac{dy}{dt} = y(y - 1)$



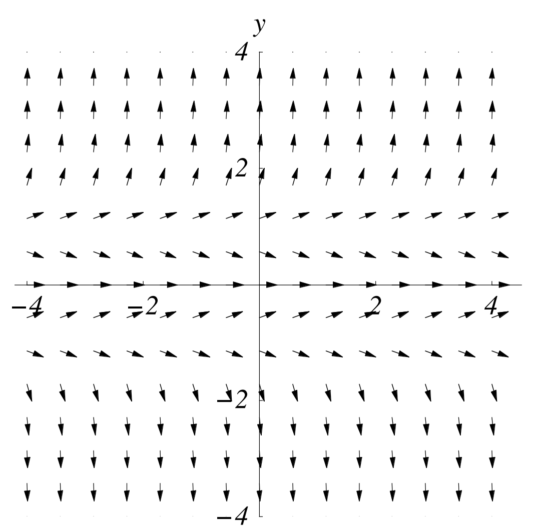
(i)



(ii)



(iii)



(iv)

Solution.

(i) — (h)	(ii) — (d)	(iii) — (b)	(iv) — (c)

7. (8 points) Solve the initial-value problem

$$\begin{aligned} y'' + 2y' + 2y &= 0 \\ y(0) &= 2 \\ y'(0) &= 3 \end{aligned}$$

Solution. The characteristic equation is $r^2 + 2r + 2 = 0$, which has roots $r = -1 \pm i$. Thus, a general solution is

$$y = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t = e^{-t}(c_1 \cos t + c_2 \sin t)$$

Since

$$y' = -e^{-t}(c_1 \cos t + c_2 \sin t) + e^{-t}(-c_1 \sin t + c_2 \cos t),$$

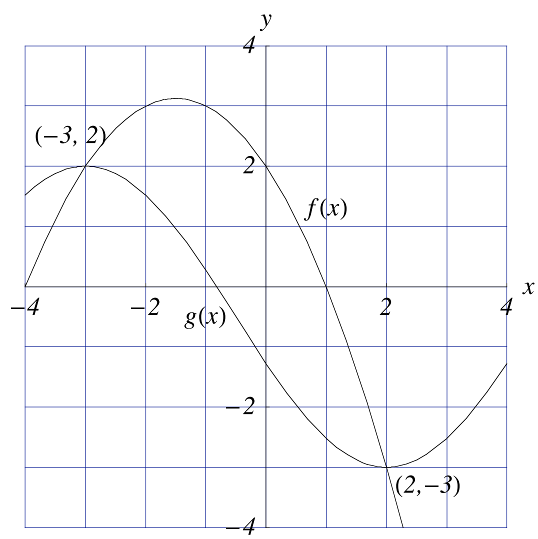
we know that

$$\begin{aligned} 2 &= y(0) = c_1 \\ 3 &= y'(0) = -c_1 + c_2 \end{aligned}$$

and $c_1 = 2$ and $c_2 = 5$. Therefore, the solution to our initial-value problem is

$$y = e^{-t}(2 \cos t + 5 \sin t).$$

8. (9 points) Let R be the region bounded by the curves $y = f(x)$ and $y = g(x)$ shown in the graph below.



(a) Write a definite integral that will give the area of the region R.

(b) Write a definite integral that will give the volume of the solid generated when the region R is revolved about the horizontal line $y = -5$.

(c) If the base of a solid V is the region R and the cross-sections of the solid perpendicular to the x-axis are squares, write a definite integral that will give the the volume of V.

Solution.

(a) $\displaystyle\int_{-3}^{2} f(x) - g(x)\, dx$

(b) $\int_{-3}^{2} \pi(g(x)+5)^2 - \pi(f(x)+5)^2\,dx$

(c) $\int_{-3}^{2} (f(x)-g(x))^2\,dx$

9. (12 points) Consider the system

$$\frac{dx}{dt} = x(2-x-y)$$
$$\frac{dy}{dt} = y(y-x).$$

(a) Find the x and y-nullclines of the system. You will be asked to graph the nullclines in part (d) of this problem.

(b) Find all of the equilibrium solutions. You will be asked to graph the equilibrium solutions in part (d) of this problem.

(c) Sketch and *label* the nullclines on the graph below. Be sure to indicate the direction of the solution on the nullclines. Sketch the trajectory in xy-plane that begins at $(2,1)$.

Solution.

(a) The x-nullclines are

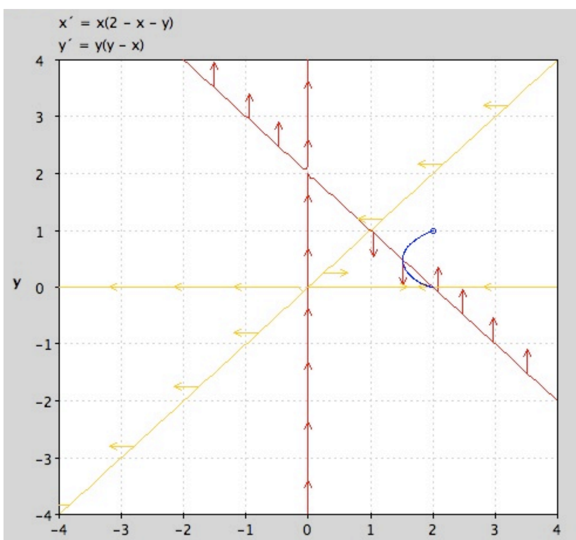
$$x = 0$$
$$y = -x+2.$$

The y-nullclines are

$$y = 0$$
$$y = x.$$

(b) The equilibrium solutions are $(0,0)$, $(1,1)$, and $(2,0)$.

(c)



10. (8 points) A tank initially holds 100 gal of pure water. At time $t=0$, a solution containing 2 lb of salt per gallon begins to enter the tank at a rate of 3 gallons per minute. At the same time a drain is opened at the bottom of the tank sol that the volume of the solution in the tank remains constant.

(a) Write down a differential equation for $x(t)$, the amount of salt in the dispenser at time t. Be sure to include your initial condition.

(b) Solve the differential equation from part (a).

Solution.

(a) $\dfrac{dx}{dt}$ = rate in − rate out $= 6 - \dfrac{3x}{100}$. The initial condition is $x(0) = 0$

(b) The differential equation may be solved as a first-order linear differential equation or by using separation of variables,

$$x(t) = 200 - 200e^{-3t/100}.$$

11. (6 points) Suppose we know that $\sum_{n=1}^{\infty} a_n$ converges to 0.8. We are given *no* other information about the infinite series. For each of the following statements circle

- *True* if the statement *must* be true,
- *False* if the statement *must* be false, and
- *Inconclusive* if the statement could be either true or false.

You do not need to justify your answers.

(a) $\lim_{n\to\infty} a_n = 0.8$	*True*	*False*	*Inconclusive*
(b) $\lim_{n\to\infty} a_n = 0$	*True*	*False*	*Inconclusive*
(c) $\lim_{n\to\infty} \dfrac{\lvert a_{n+1}\rvert}{\lvert a_n\rvert} > 1$	*True*	*False*	*Inconclusive*
(d) $a_{n+1} < a_n$ for all n	*True*	*False*	*Inconclusive*
(e) $\lim_{n\to\infty} \dfrac{1}{\lvert a_n\rvert} = \infty$	*True*	*False*	*Inconclusive*
(f) $\lim_{n\to\infty} S_n = 0.8$, where $S_n = a_1 + a_2 + \cdots + a_n$.	*True*	*False*	*Inconclusive*

Solution. (a) *False*, (b) *True*, (c) *False*, (d) *Inconclusive*, (e) *True*, (f) *True.*