Math 131 Exam 1

1. Find the average rate of change of $y = x^2 + x + 1$ over the interval $[0, 1]$.

(A) 0
(B) 1
✓ (C) 2
(D) 3
(E) 4
(F) 5
(G) 6
(H) -1
(I) None of the above
(J) Undecidable

$$\frac{y(1) - y(0)}{1 - 0} = \frac{3-1}{1} = 2$$

2. Find $\lim_{x\to 1} \frac{x^2+5}{x^3-x^2+10}$.

(A) 0
(B) 0.1
(C) 0.2
(D) 0.3
(E) 0.4
(F) 0.5
✓(G) 0.6
(H) 0.7
(I) 0.8
(J) 0.9

$$= \frac{1+5}{1-1+10} = \frac{6}{10} = 0.6$$

3. Find $\lim_{x\to 1} \frac{x^2-1}{\sqrt{x^2+15}-4}$.

(A) 9
✓(B) 8
(C) 7
(D) 6
(E) 5
(F) 4
(G) 3
(H) 2
(I) 1
(J) 0

$$= \lim_{x\to 1} \frac{(x^2-1)(\sqrt{x^2+15}+4)}{(\sqrt{x^2+15}-4)(\sqrt{x^2+15}+4)}$$

$$= \lim_{x\to 1} \frac{(x^2-1)(\sqrt{x^2+15}+4)}{(x^2+15)-16}$$

$$= \lim_{x\to 1} \frac{(x^2-1)(\sqrt{x^2+15}+4)}{x^2-1}$$

$$= \lim_{x\to 1} \sqrt{x^2+15}+4 = \sqrt{16}+4 = 8$$

4. Find $\lim_{x\to 0} \frac{\tan(2x)}{\tan(4x)}$.

(A) $2/\pi$
(B) ∞
(C) 2π
✓(D) 0.5
(E) 2
(F) 4
(G) 0.25π
(H) 8π
(I) 0.125
(J) 10

$$= \lim_{x\to 0} \frac{\frac{\sin 2x}{\cos 2x}}{\frac{\sin 4x}{\cos 4x}} = \lim_{x\to 0} \frac{\sin 2x}{\sin 4x} \cdot \frac{\cos 4x}{\cos 2x}$$

$$= \lim_{x\to 0} \frac{\sin 2x}{\sin 4x} \cdot \underbrace{\lim_{x\to 0} \frac{\cos 4x}{\cos 2x}}_{= \frac{1}{1} = 1}$$

$$= \lim_{x\to 0} \frac{\sin 2x}{\sin 4x}$$

$$= \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{2x}{4x} \cdot \frac{4x}{\sin 4x}$$

$$= \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \lim_{x\to 0} \frac{1}{2} \cdot \lim_{x\to 0} \frac{4x}{\sin 4x}$$

$$= 1 \cdot \frac{1}{2} \cdot 1 = 0.5$$

5. What is the x-intercept of the tangent line to the curve $y = 4 - x^2$ at the point $(-1, 3)$?

(A) 3
(B) 0.3
(C) -2
(D) 0.5
(E) 0.8
✓(F) -2.5
(G) 0.7
(H) -1
(I) 0
(J) ∞

slope of the tangent line to $y = 4 - x^2$ at $x = -1$ is

$$\lim_{x \to -1} \frac{y(x) - y(-1)}{x - (-1)}$$

$$= \lim_{x \to -1} \frac{(4 - x^2) - 3}{x + 1} = \lim_{x \to -1} \frac{1 - x^2}{x + 1} = \lim_{x \to -1} \frac{(1+x)(1-x)}{1+x}$$

$$= \lim_{x \to -1} (1 - x) = 2$$

The point-slope formula gives the tangent line equation

$$y - 3 = 2(x + 1)$$

Setting $y = 0$, we get $-3 = 2x + 2$ $\therefore -5 = 2x$

$\therefore$ the x-intercept is $x = -2.5$.

6. What is $\lim_{x\to\infty} \frac{x+2\sin(x)+\sqrt{x}}{3x-10\cos(x)}$?

(A) 0
(B) 1
(C) 2
(D) 1/2
(E) 3
✓(F) 1/3
(G) -4
(H) 1/4
(I) -5
(J) 1/5

$$= \lim_{x\to\infty} \frac{x+2\sin x+\sqrt{x} \;/\; x}{3x-10\cos x \;/\; x}$$

$$= \lim_{x\to\infty} \frac{1+2\frac{\sin x}{x}+\frac{\sqrt{x}}{x}}{3-10\frac{\cos x}{x}}$$

$$= \frac{1+2\lim_{x\to\infty}\frac{\sin x}{x}+\lim_{x\to\infty}\frac{\sqrt{x}}{x}}{3-10\lim_{x\to\infty}\frac{\cos x}{x}}$$

$$= \frac{1+0+0}{3-0} = \frac{1}{3}$$

7. The rational function $\frac{8x^4+x^3}{2x^3+9}$ has an oblique asymptote of the form $y = ax + b$. What is $a + b$?

(A) -4
(B) 2
(C) ∞
(D) 8/9
(E) -2
✓(F) 4.5
(G) -4.5
(H) 0.5
(I) 9
(J) -9

$$
\begin{array}{r|l}
 & 4x + \frac{1}{2} \\
2x^3 + 9 & 8x^4 + x^3 \\
 & -)\ 8x^4 \qquad + 36x \\
\hline
 & x^3 \quad - 36x \\
 & -)\ x^3 \qquad + \frac{9}{2} \\
\hline
 & -36x - \frac{9}{2}
\end{array}
$$

$\therefore y = 4x + \frac{1}{2}$ is the asymptote.

$\therefore \quad a + b = 4 + \frac{1}{2} = 4.5$

8. The function in Problem 7 has a vertical asymptote at $x = a$. Suppose for a real number x we define $[x]$ to be the greatest integer less than or equal to x. Then what is $[a]$ equal to?

(A) -9
(B) -8
(C) -7
(D) -6
(E) -5
(F) -4
(G) -3
✓(H) -2
(I) -1
(J) 0

$$\frac{8x^4+x^3}{2x^3+9}$$

The zero of $2x^3+9$ is a

$2x^3+9=0 \quad \therefore \quad x^3 = -\frac{9}{2}$

$\therefore \quad x = -\sqrt[3]{4.5} \doteq -1.\ldots$

$\therefore \quad [x] = -2$

9. Let $f(t) = t^2 - t + 1$. Find $\lim_{x\to 100^-} f([x])$. (Note: $[x]$ is defined in Problem 8.)

(A) -2134
(B) 2134
(C) -4001
(D) 4001
(E) -6379
(F) 6379
(G) -8426
(H) 8426
(I) -9703
✓(J) 9703

$$= f\left(\lim_{x\to 100^-}[x]\right) = f(99)$$

$$= 99^2 - 99 + 1 = 9703$$

10. Let the function f be defined as $f(x) = x^2$ for $x \leq 1$ and $f(x) = ax + 3$ for $x > 1$. What is the value of a for f to be continuous?

(A) 1
(B) -1
✓(C) -2
(D) 1/2
(E) 3
(F) -1/3
(G) 4
(H) -1/4
(I) 5
(J) -1/5

$f(1) = 1^2 = 1$.

$$\lim_{x \to 1^+} f(x) = f(1) = 1$$ for f to be continuous at x=1.

$$\lim_{x \to 1^+} f(x) = a+3$$

$\therefore$ $a + 3 = 1$

$\therefore$ $a = -2$.

11. Suppose $\lim_{x\to -4}(x \lim_{x\to 0} f(x)) = -100$. What is $\frac{1}{\lim_{x\to 0} f(x)}$?

(A) Undecidable
(B) 0.01
(C) 0.02
(D) 0.03
✓(E) 0.04
(F) 0.05
(G) 0.06
(H) 0.07
(I) 0.08
(J) 0.09

$$\|$$

$$-4 \lim_{x\to 0} f(x) = -100$$

$$\therefore \lim_{x\to 0} f(x) = 25$$

$$\therefore \frac{1}{\lim_{x\to 0} f(x)} = \frac{1}{25} = 0.04$$

12. The function $3x - \sqrt{9x^2 + 18x + 2}$ has a horizontal asymptote at $y = a$. What is a equal to?

(A) 4
(B) 3
(C) 2
(D) 1
(E) 0
(F) -1
(G) -2
✓(H) -3
(I) -4
(J) 0.5

$$\lim_{x\to\infty}\left(3x - \sqrt{9x^2+18x+2}\right)$$

$$= \lim_{x\to\infty} \frac{\left(3x - \sqrt{9x^2+18x+2}\right)\left(3x+\sqrt{9x^2+18x+2}\right)}{3x+\sqrt{9x^2+18x+2}}$$

$$= \lim_{x\to\infty} \frac{(3x)^2 - \left(\sqrt{9x^2+18x+2}\right)^2}{3x+\sqrt{9x^2+18x+2}}$$

$$= \lim_{x\to\infty} \frac{9x^2 - (9x^2+18x+2)}{3x + 3x\sqrt{1+\frac{18x}{9x^2}+\frac{2}{9x^2}}}$$

$$= \lim_{x\to\infty} \frac{-18x-2}{3x+3x\sqrt{1+\frac{2}{x}+\frac{2}{9x^2}}}$$

$$= \lim_{x\to\infty} \frac{-18+\frac{-2}{x}}{3+3\sqrt{1+\frac{2}{x}+\frac{2}{9x^2}}}$$

$$= \frac{-18}{3+3} = -3$$

13. Find $\lim_{x\to 0} \frac{\sin(1-\cos(x))}{x^2}$. (Hint: Use the formula $1-\cos(2t) = 2(\sin(t))^2$.)

(A) ∞
(B) 0
(C) -1.5
(D) 1.5
(E) $-\pi/2$
(F) $\pi/2$
(G) -1
(H) 1
(I) -0.5
✓(J) 0.5

$$= \lim_{x\to 0} \frac{\sin(\overbrace{1-\cos x}^{\theta})}{\underbrace{1-\cos x}_{\theta}} \cdot \frac{1-\cos x}{x^2}$$

$$= \lim_{\cancel{x}\to 0} \frac{\sin\theta}{\theta} \cdot \lim_{x\to 0} \frac{1-\cos x}{x^2}$$

$\theta \to 0$ if $x \to 0$

$$= \underbrace{\lim_{\theta\to 0} \frac{\sin\theta}{\theta}}_{1} \cdot \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{x^2}$$

$$= 2\lim_{x\to 0} \frac{(\sin\frac{x}{2})^2}{x^2} = 2\lim_{x\to 0} \left(\frac{\sin\frac{x}{2}}{x}\right)^2$$

$$= 2^{-1} \lim_{x\to 0} \left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$

$$= 2^{-1} \left(\lim_{x\to 0} \frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = 2^{-1} \cdot \left(\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\right)^2$$

$$= 2^{-1}.$$

(Set $\theta = \frac{x}{2}$. $\theta \to 0$ as $x \to 0$)

14. Let $f(x) = \frac{x^2-25}{x+5}$. Though the function is not defined at $x = a$ for a certain a, it can be made continuous if we define $f(a) = b$ for a certain b. What is b/a?

(A) 1
✓(B) 2
(C) 3
(D) 4
(E) 5
(F) 6
(G) 7
(H) 8
(I) 9
(J) 10

$$\frac{x^2-25}{x+5} = \frac{(x-5)(x+5)}{x+5} = x-5$$

if $x \neq -5$

At $x = -5$, $f(x)$ is not defined because the denominator of $f(x)$ is zero.

$\therefore\ a = -5$

$$\lim_{x \to -5} f(x) = \lim_{x \to -5} x-5 = -10$$

$\therefore\ -10 = f(a)$ to make f continuous.

$\therefore\ b = -10$

$\therefore\ b/a = -10/-5 = 2$

15. Let $f(x)$ and $g(x)$ be two functions defined on the real line. Consider the following three statements.

T (I) If $\lim_{x\to 0} f(x)$ exists and $\lim_{x\to 0} g(x)$ exists, then $\lim_{x\to 0}(f(x)+g(x))$ exists.

F (II) If $\lim_{x\to 0} f(x)$ exists and $\lim_{x\to 0} g(x)$ exists, then $\lim_{x\to 0} \frac{1}{f(x)+g(x)}$ exists.

T (III) If $\lim_{x\to 0} f(x)$ exists and $\lim_{x\to 0}(f(x)+g(x))$ exists, then $\lim_{x\to 0} g(x)$ exists.

Let us denote a true statement by T and a false statement by F. Which one of the following is the correct answer for the truth or falsity of each statement arranged in order?

(A) (T,T,T)
(B) (T,T,F)
✓ (C) (T,F,T)
(D) (T,F,F)
(E) (F,T,T)
(F) (F,T,F)
(G) (F,F,T)
(H) (F,F,F)
(I) None of the above
(J) Undecidable

(II) fails when $\lim_{x\to 0}\left(f(x)+g(x)\right) = 0$.

16. Let $f(x)$ and $g(x)$ be two functions defined on the real line. Consider the following three statements.

F (I) If $\lim_{x\to 0} f(x)$ exists and $\lim_{x\to 0} g(x)$ exists, then $\lim_{x\to 0} f(x)/g(x)$ exists.

F (II) If $\lim_{x\to 0} f(x)$ does not exist and $\lim_{x\to 0} g(x)$ does not exist, then $\lim_{x\to 0} f(x)/g(x)$ does not exist.

T (III) If $f(x)$ and $g(x)$ are continuous functions, then $f(g(x))$ is a continuous function.

Let us denote a true statement by T and a false statement by F. Which one of the following is the correct answer for the truth or falsity of each statement arranged in order?

(A) (T,T,T)
(B) (T,T,F)
(C) (T,F,T)
(D) (T,F,F)
(E) (F,T,T)
(F) (F,T,F)
✓(G) (F,F,T)
(H) (F,F,F)
(I) None of the above
(J) Undecidable

(I) fails when $\lim_{x\to 0} g(x) = 0$.

(II) fails when, for instance, $f = g$.

17. A function $f(x)$ satisfies $\underbrace{1 - x^2/2}_{g(x)} \le f(x) \le \underbrace{1 - x^2/2 + x^4/24}_{h(x)}$. Find $\lim_{x\to 0} f(x)$.

(A) 10
(B) 9
(C) 8
(D) 7
(E) 6
(F) 5
(G) 4
(H) 3
(I) 2
✓ (J) 1

$$\lim_{x\to 0} g(x) = 1 = \lim_{x\to 0} h(x)$$

$$\therefore \lim_{x\to 0} f(x) = 1$$ by the Sandwich Theorem

18. Suppose $\lim_{x\to 1}(f(x)+g(x)) = 5$ and $\lim_{x\to 1}(f(x)-g(x)) = 1$. What is $\lim_{x\to 1}(f(x)g(x))$ equal to?

(A) 10
(B) 9
(C) 8
(D) 7
✓(E) 6
(F) 5
(G) 4
(H) 3
(I) 2
(J) 1

Set $u(x) = f(x) + g(x)$ $\quad \lim_{x\to 1} u(x) = 5$

$v(x) = f(x) - g(x)$ $\quad \lim_{x\to 1} v(x) = 1$

$$f(x) = \frac{u(x)+v(x)}{2}$$

$$g(x) = \frac{u(x)-v(x)}{2}$$

$$\therefore \lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{u(x)+v(x)}{2}$$

$$= \frac{\lim_{x\to 1} u(x) + \lim_{x\to 1} v(x)}{2} = \frac{5+1}{2}$$

$$= 3$$

$$\lim_{x\to 1} g(x) = \lim_{x\to 1} \frac{u(x)-v(x)}{2}$$

$$= \frac{\lim_{x\to 1} u(x) - \lim_{x\to 1} v(x)}{2} = \frac{5-1}{2} = 2$$

$$\therefore \lim_{x\to 1} f(x)g(x) = \lim_{x\to 1} f(x) \lim_{x\to 1} g(x)$$

$$= 3 \cdot 2 = 6$$

Name: ID:

19. Show that the polynomial $f(x) = -x^3 + x + 1$ has a root between 1.3 and 1.4.

$$f(1.3) = -(1.3)^3 + 1.3 + 1$$
$$= -2.197 + 2.3 > 0$$
$$f(1.4) = -(1.4)^3 + 1.4 + 1$$
$$= -2.744 + 2.4 < 0$$

$\therefore$ There is a root between 1.3 and 1.4 by the ~~Mean~~ Intermediate Value Theorem

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20. Tom rowed a kayak from town A to town B at a speed of 5 miles per hour. Then he biked from town B to town C at a speed of 10 miles per hour for two hours. His travel distance totaled 50 miles. What is his average speed (in miles per hour)?

distance between town B & town C

= 20 miles

$\therefore$ distance between town A & town B

= 30 miles

Tom kayaked for 6 hours.

$\therefore$ Ave. speed = $\dfrac{\text{total distance covered}}{\text{total lapse of time}}$

$= \dfrac{50}{6+2} = \dfrac{50}{8} = 6.25\ m/h$