1. The function $y = \frac{x+3}{x-2}$ has a vertical asymptote at $x = A$ and a horizontal asymptote at $y = B$. What is $A + B$?

(A) 0
(B) 1
(C) 2
✓ (D) 3
(E) 4
(F) 5
(G) 6
(H) 7
(I) 8
(J) 9

$A = 2$

$B = 1$ since $\lim_{x \to \pm\infty} \frac{x+3}{x-2} = 1$

$\therefore A + B = 3$

2. $f(x) = x^3 + x + 5$. Find $(f^{-1})'(5)$, where f^{-1} denotes the inverse function to f.

(A) 5
(B) 4
(C) 3
(D) 2
✓ (E) 1
(F) 10
(G) 9
(H) 8
(I) 7
(J) 6

$f'(x) = 3x^2+1 > 0$

The function is strictly increasing, so that f has an inverse f^{-1}.

$f(0) = 5$, $\therefore f^{-1}(5) = 0$

$\therefore (f^{-1})'(5) = \dfrac{1}{f'(0)} = \dfrac{1}{1} = 1$

3. A particle moves on the curve $x^2 + xy - y^3 = 1$. At the point $(x, y) = (1, 1)$ you observe that $\frac{dx}{dt} = 1$. Find $\frac{dy}{dt}$ at the same point. (t is the time parameter.)

(A) 0.5
(B) 1
✓ (C) 1.5
(D) 2
(E) 2.5
(F) 3
(G) 3.5
(H) 4
(I) 4.5
(J) 5

$$\frac{d}{dt}(x^2+xy-y^3) = \frac{d1}{dt} = 0$$

$$2x\frac{dx}{dt} + \frac{dx}{dt}y + x\frac{dy}{dt} - 3y^2\frac{dy}{dt} = 0$$

$$x=1,\ y=1,\ \frac{dx}{dt}=1.$$

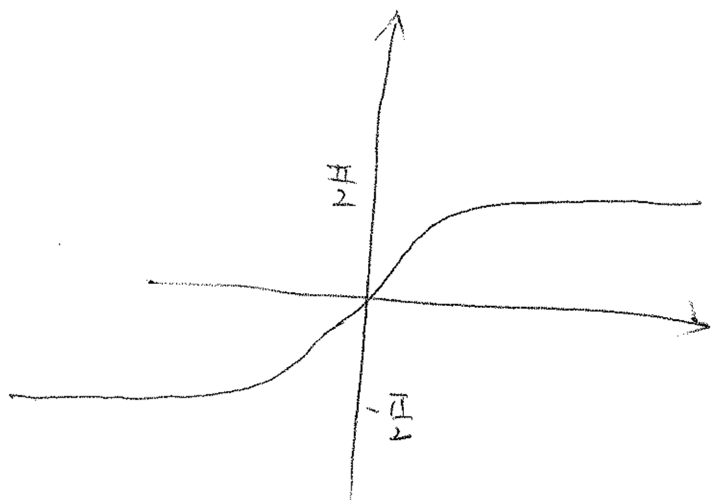
$$\therefore\ 2+1+y'-3y'=0$$

$$\therefore\ 3 = 2y'$$

$$\therefore\ y' = \frac{3}{2}$$

4. Find $\lim_{x\to\infty} \tan^{-1}(x)$.

(A) 5π
(B) 4.5π
(C) 4π
(D) 3.5π
(E) 3π
(F) 1.5π
(G) 2π
✓ (H) 0.5π
(I) π
(J) 0



5. Let $f(x) = 3x^5 - 5x^4 + 1$ be defined over the entire real line. How many points of inflection does $f(x)$ have?

(A) 4
(B) 3
(C) 2
✓(D) 1
(E) 0
(F) 9
(G) 8
(H) 7
(I) 6
(J) 5

$$f' = 15x^4 - 20x^3$$

$$f'' = 60x^3 - 60x^2$$

$$= 60x^2(x-1)$$

f'' changes sign only around $x = 1$

$\therefore$ $x=1$ is the only point of inflection.

6. $f(x) = x\ln(x)$, where its domain is $x > 0$. Let $x = A$ be its critical point. What is A equal to?
(A) 0
(B) e
✓ (C) e^{-1}
(D) e^3
(E) e^{-3}
(F) e^2
(G) e^{-2}
(H) 1
(I) $e^{1/2}$
(J) The function has no critical points in its domain.

$$0 = f' = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

$$\therefore \ln x = -1$$

$$\therefore x = e^{-1}$$

By L'HOPITAL

7. Find $\lim_{x\to 0} \frac{\sin(x)-x+\frac{x^3}{6}}{x^5}$.

(A) 1
(B) 1/5
(C) 1/30
(D) 1/60
(E) 1/90
✓ (F) 1/120
(G) 1/150
(H) 1/180
(I) 1/210
(J) Undecidable

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$$

$$= \lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{5x^4}$$

$$= \lim_{x\to 0} \frac{-\sin x + x}{20x^3}$$

$$= \lim_{x\to 0} \frac{-\cos x + 1}{60x^2}$$

$$= \lim_{x\to 0} \frac{\sin x}{120x}$$

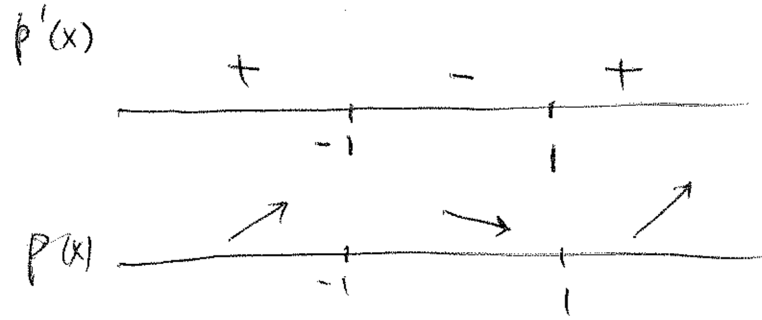
$$= \lim_{x\to 0} \frac{\cos x}{120}$$

$$= \frac{1}{120}$$

8. Let $p(x) = x^5 - 5x - 5$. The first derivative test can establish that $p(x) = 0$ has exactly one solution, and this solution is between two consecutive integers n and $n+1$. Set $x_0 = n+1$. Using Newton's method we can find $x_1, x_2, x_3, \cdots$ to estimate the solution. What is x_1 equal to?

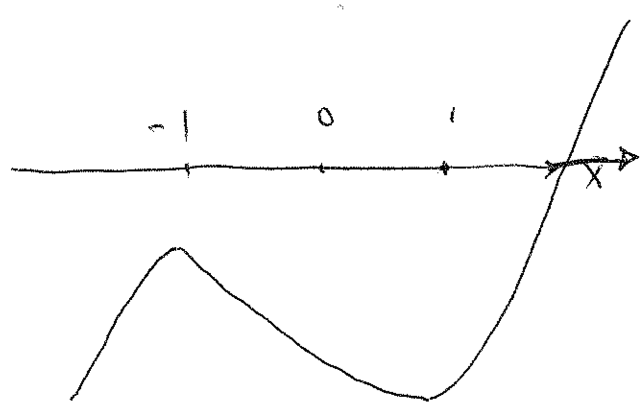
(A) 0.5
(B) 71/8
(C) -3.5
(D) -531/90
(E) -6.5
(F) -303/100
(G) 1.5
(H) 23/37
✓ (I) 133/75
(J) 524/63

$$p'(x) = 5x^4 - 5 = 5(x^4 - 1)$$
$$= 5(x^2+1)(x^2-1) = 5(x^2+1)(x+1)(x-1)$$



$p(-1) = -1$, $p(1) = -9$

Graph of p



$p(1) = -9 < 0$

$p(2) = \cancel{25}\ 32 - 10 - 5 = 17 > 0$

∴ The root is between 1 & 2. ($n = 1$)

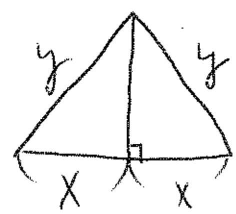
$X_0 = 2$

$$X_1 = X_0 - \frac{p(X_0)}{p'(X_0)} = 2 - \frac{17}{75}$$

$$= \frac{133}{75}$$

9. You have a string 6 meters long. You want to use it to enclose a *isosceles* triangle of maximum area. What is the area of the triangle (in square meters)?

(A) 2
(B) $1/2\sqrt{3}$
(C) $1/\sqrt{2}$
(D) $3\sqrt{2}$
(E) $3/\sqrt{2}$
(F) $2\sqrt{2}$
✓(G) $\sqrt{3}$
(H) $2\sqrt{2}$
(I) $2\sqrt{3}$
(J) 3



$$2x + 2y = 6$$
$$x + y = 3$$

$$\text{Area} = x\sqrt{y^2 - x^2} = x\sqrt{(3-x)^2 - x^2} = x\sqrt{9-6x}$$

More convenient to maximize $(\text{Area})^2$

$$= x^2(9-6x) = 9x^2 - 6x^3 = f(x)$$

$$0 = f'(x) = 18x - 18x^2$$

$\therefore x = 1$ $\qquad \therefore y = 2$

$$\therefore \text{Area} = 1 \cdot \sqrt{2^2 - 1} = \sqrt{3}$$

10. Let $F(x)$ be the antiderivative of $3x^2+2x+1$ with $F(2)=0$. What is $F(1)$ equal to?

(A) -2
(B) -3
(C) -4
(D) -5
(E) -6
(F) -7
(G) -8
(H) -9
(I) -10
✓(J) -11

$F(x) = \int (3x^2+2x+1)\,dx$

$= x^3 + x^2 + x + C$

$0 = F(2) = 8+4+2+C \qquad \therefore C = -14$

$F(x) = x^3 + x^2 + x - 14$

$\therefore F(1) = 3 - 14 = -11$

11. Find $\sum_{k=1}^{3} k(2k+1)$.

(A) Undecidable
(B) 38
(C) 37
(D) 36
(E) 35
✓(F) 34
(G) 33
(H) 32
(I) 31
(J) 30

$= 1 \cdot 3 + 2 \cdot 5 + 3 \cdot 7$

$= 3 + 10 + 21$

$= 34$

12. Find the area of the region bounded between the curve $y = \frac{1}{\sqrt{1-x^2}}$, the x-axis, and the vertical lines $x = 0$ and $x = \frac{1}{2}$.

(A) $\pi/4$
(B) $-\pi/4$
✓(C) $\pi/6$
(D) $-\pi/6$
(E) $\pi/3$
(F) $-\pi/3$
(G) $\pi/2$
(H) $-\pi/2$
(I) π
(J) 2π

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1}(x) \Big|_0^{\frac{1}{2}}$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)$$

$$= \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

13. Find the average value of $f(x) = 2x + \frac{1}{x}$ over the interval $[1, e]$.

(A) $(e+1)/(e-1)$
(B) $e^2-1)/(e^2+1)$
✓(C) $e^2/(e-1)$
(D) $e/(e^2-1)$
(E) 1
(F) e
(G) e^{-1}
(H) $1+e$
(I) $e-1$
(J) $e^3/(e^3+1)$

$$\frac{\int_1^e \left(2x+\frac{1}{x}\right) dx}{e-1}$$

$$= \frac{x^2 + \ln x \Big|_1^e}{e-1}$$

$$= \frac{(e^2+\ln e) - (1^2 + \ln 1)}{e-1}$$

$$= \frac{(e^2+1) - (1+0)}{e-1}$$

$$= \frac{e^2}{e-1}$$

14. Find $\lim_{x\to 0} \frac{\csc(x)-\cot(x)}{x}$.

(A) -0.5
(B) -1
(C) -1.5
(D) -2
(E) 2
(F) 1.5
(G) 1
✓ (H) 0.5
(I) 0
(J) Undecidable

$$= \lim_{x\to 0} \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x}$$

$$= \lim_{x\to 0} \frac{\frac{1-\cos x}{\sin x}}{x}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{x \sin x}$$

L'HOPITAL

$$= \lim_{x\to 0} \frac{\sin x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0} \frac{\cos x}{\cos x + \cos x - x\sin x}$$

$$= \frac{1}{2}$$

15. Consider the following three statements.

(I) $F(x)$ is an antiderivative of $f(x)$ over the real line if $F'(x) = f(x)$ for all x on the real line.
(II) Two antiderivatives of $f(x)$ over an open interval (a, b) differ by a constant over the interval.
(III) Newton's method is a scheme that can always result in an estimate as close to the actual solution as we wish by carrying out enough steps.

Let us denote a true statement by T and a false statement by F. Which one of the following is the correct answer for the truth or falsity of each statement arranged in order?

(A) (T,T,T)
(B) (T,T,F)
(C) (T,F,T)
(D) (T,F,F)
(E) (F,T,T)
(F) (F,T,F)
(G) (F,F,T)
(H) (F,F,F)
(I) None of the above
(J) Undecidable

(III) is false. x_0 ~~has~~ must be chosen appropriately.

16. Consider the following three statements.

(I) $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} k^9}{n^{10}} = 1$.
(II) Let $f(x)$ be continuous over the interval $[a, b]$. Then the area function $F(x) = \int_a^x f(x)dx$ is an antiderivative of $f(x)$ over $[a, b]$.
(III) $\int_{-1}^{1} |x|dx = 2$.

Let us denote a true statement by T and a false statement by F. Which one of the following is the correct answer for the truth or falsity of each statement arranged in order?

(A) (T,T,T)
(B) (T,T,F)
(C) (T,F,T)
(D) (T,F,F)
(E) (F,T,T)
✓ (F) (F,T,F)
(G) (F,F,T)
(H) (F,F,F)
(I) None of the above
(J) Undecidable

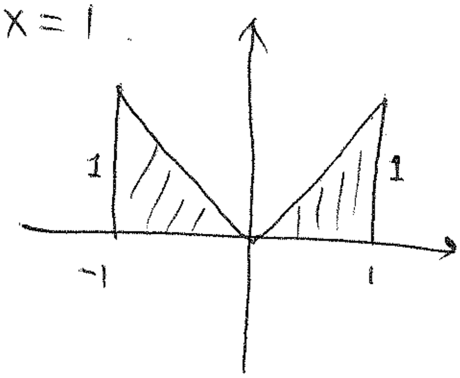
(I) is false

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^9}{n^{10}} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \left(\frac{k}{n}\right)^9}{n}$$

$$= \int_0^1 x^9 dx = \frac{x^{10}}{10}\Big|_0^1 = \frac{1}{10}.$$

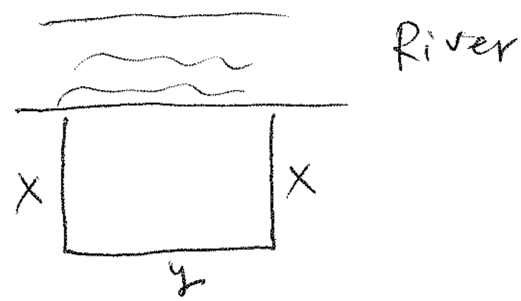
(III) is false $\int_{-1}^{1} |x| dx$ is the area underneath $y = |x|$, above the x-axis & between $x=-1$ & $x=1$.

So $\int_{-1}^{1} |x| dx = 1$.



17. A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 1000 meters of wire at your disposal, what is the largest area (in thousand square meters) you can enclose?

✓(A) 125
(B) 100
(C) 75
(D) 50
(E) 25
(F) 35
(G) 45
(H) 55
(I) 65
(J) 85



Maximize

$$2x + y = 1000$$

$$f = xy$$
$$= x(1000 - 2x)$$
$$= 1000x - 2x^2$$

$$0 = f' = 1000 - 4x$$

$$\therefore x = 250$$
$$y = 500$$

$$\therefore \quad xy = 250 \cdot 500 = 125{,}000$$

18. $f(x) = \int_0^x \tan^{-1}(x)dx$. Find $f''(2)$.

(A) 0.6
(B) 0.5
(C) 0.4
(D) 0.3
✓(E) 0.2
(F) 0.1
(G) 1
(H) 2
(I) 3
(J) 4

$$f'(x) = \tan^{-1}(x)$$

$$f''(x) = \frac{1}{1+x^2}$$

$$f''(2) = \frac{1}{1+2^2} = \frac{1}{5}$$

19. Find $\lim_{x\to 0}(e^x + x)^{1/x}$.

(A) Undecidable
(B) 0
(C) 1/2
(D) 1
(E) e^{-3}
(F) e^3
(G) e^{-1}
(H) e
(I) e^{-2}
✓ (J) e^2

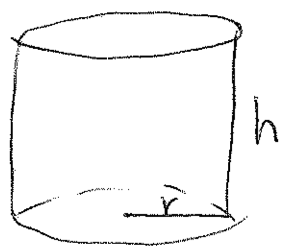
$$= \lim_{x\to 0} e^{\frac{1}{x}\ln(e^x+x)}$$

$$= e^{\lim_{x\to 0} \frac{\ln(e^x+x)}{x}}$$

$$= e^{\lim_{x\to 0} \frac{\frac{e^x+1}{e^x+x}}{1}} = e^2$$

20. Find the minimum surface area (in square centimeters) of a beer can if its volume is kept at 16π cubic centimeters. (The beer can is in the shape of a right circular cylinder. Its surface area is the sum of the lateral, the top and the bottom base areas. We ignore the thickness of the can surface.)

(A) 16π
(B) 18π
(C) 14π
(D) 12π
(E) 20π
(F) 36π
(G) 24π
(H) 64π
(I) 32π
(J) Undecidable



$$\text{Vol} = \pi r^2 h = 16\pi$$
$$r^2 h = 16$$
$$h = \frac{16}{r^2}$$

minimize surface area

$$f = 2\pi r h + 2\pi r^2$$
$$= 2\pi r \cdot \frac{16}{r^2} + 2\pi r^2$$
$$= \frac{32\pi}{r} + 2\pi r^2$$

$$0 = f' = -\frac{32\pi}{r^2} + 4\pi r$$
$$\frac{\overset{8}{\cancel{32\pi}}}{r^2} = \cancel{4\pi} r$$
$$\therefore\ 8 = r^3$$
$$\therefore\ 2 = r$$
$$\therefore\ h = \frac{16}{r^2} = 4$$
$$\therefore\ \text{surface area} = 2\pi r h + 2\pi r^2$$
$$= 2\pi \cdot 2 \cdot 4 + 2\pi \cdot 2^2 = 16\pi + 8\pi$$
$$= 24\pi.$$