EXAM 1 **MAT 167 Calculus I** Spring 2011

**Name:**____________________ Section:__________

IMPORTANT: Each answer must include either supporting work or an explanation of your reasoning. These elements are considered to be the main part of each answer and will be graded.

1. (40 pts) Evaluate each of the following limits.

(a) $\lim_{x\to 4}(5x^2 - 2x + 3)$

**Solution:** $\lim_{x\to 4}(5x^2 - 2x + 3) = 5(4)^2 - 2\cdot 4 + 3 = 75.$

(b) $\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - 3x + 2}$

**Solution:**

$$\begin{aligned}\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - 3x + 2} &= \lim_{x\to 1} \frac{(x+2)(x-1)}{(x-1)(x-2)} \\ &= \lim_{x\to 1} \frac{x+2}{x-2} \\ &= -3.\end{aligned}$$

(c) $\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1}$

**Solution:**

$$\begin{aligned}\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1} &= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x+1)(x-1)} \\ &= \lim_{x\to 1} \frac{x^2+x+1}{x+1} \\ &= \frac{3}{2}.\end{aligned}$$

(d) $\lim_{x \to 1} \dfrac{x-1}{\sqrt{x^2+3}-2}$

**Solution:**

$$\begin{aligned}
\lim_{x \to 1} \frac{x-1}{\sqrt{x^2+3}-2} &= \lim_{x \to 1} \frac{x-1}{\sqrt{x^2+3}-2} \cdot \frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2} \\
&= \lim_{x \to 1} \frac{(x-1)\sqrt{x^2+3}+2}{(x^2+3)-4} \\
&= \lim_{x \to 1} \frac{(x-1)\sqrt{x^2+3}+2}{x^2-1} \\
&= \lim_{x \to 1} \frac{(x-1)\sqrt{x^2+3}+2}{(x+1)(x-1)} \\
&= \lim_{x \to 1} \frac{\sqrt{x^2+3}+2}{x+1} \\
&= 2.
\end{aligned}$$

2. (10 pts) For what value of $c$ is the function

$$f(x) = \begin{cases} x^2 - 1 & \text{if} \quad x < 3, \\ 2cx & \text{if} \quad x \geq 3 \end{cases}$$

continuous on $(-\infty, \infty)$?

**Solution:** For all $x < 3$, $f(x) = x^2 - 1$ is continuous since it is a polynomial. Similarly, for all $x > 3$ $f(x) = 2cx$ is continuous no matter what the value of $c$ is. Thus to answer the question we need to determine the value of $c$ that makes $f(x)$ continuous at $x = 3$. Now

$$\begin{aligned}
\lim_{x \to 3-} f(x) &= \lim_{x \to 3-} (x^2 - 1) = 8, \\
\lim_{x \to 3+} f(x) &= \lim_{x \to 3+} 2cx = 6c = f(3).
\end{aligned}$$

In order for $f(x)$ to be continuous at $x = 3$, $8 = 6c$, that is $c = \frac{4}{3}$.

3. (10 pts) Use the Intermediate Value Theorem to show that there is a root of

$$x^3 - 3x + 1 = 0$$

in the interval $(0, 1)$.

**Solution:** $f(x) = x^3 - 3x + 1$ is continuous on the closed interval $[0, 1]$. Since $f(0) = 1 > 0$ and $f(1) = -1 < 0$, by the Intermediate Value Theorem, $x^3 - 3x + 1 = 0$ has at least one root in $(0, 1)$.

4. (20 pts) Evaluate each of the following limits.

(a) $\displaystyle\lim_{x\to\infty} \frac{3x^3 + 5x}{2x^3 - x^2 + 4}$.

**Solution:** The limit is $\frac{\infty}{\infty}$ type indeterminate form. Since the highest power of $x$ appeared in the denomenator is $x^3$,

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x^3 + 5x}{2x^3 - x^2 + 4} &= \lim_{x\to\infty} \frac{\frac{3x^3+5x}{x^3}}{\frac{2x^3-x^2+4}{x^3}} \\
&= \lim_{x\to\infty} \frac{3 + \frac{5}{x^2}}{2 - \frac{1}{x} + \frac{4}{x^3}} \\
&= \frac{3}{2}.
\end{aligned}$$

(b) $\displaystyle\lim_{x\to\infty} \frac{x + 2}{\sqrt{9x^2 + 1}}$.

**Solution:** The limit is $\frac{\infty}{\infty}$ type indeterminate form. Since the highest power of $x$ appeared in the denomenator is $x$,

$$\begin{aligned}
\lim_{x\to\infty} \frac{x + 2}{\sqrt{9x^2 + 1}} &= \lim_{x\to\infty} \frac{\frac{x+2}{x}}{\frac{\sqrt{9x^2+1}}{x}} \\
&= \lim_{x\to\infty} \frac{1 + \frac{2}{x}}{\sqrt{\frac{9x^2+1}{x^2}}} \\
&= \lim_{x\to\infty} \frac{1 + \frac{2}{x}}{\sqrt{9 + \frac{1}{x^2}}} \\
&= \frac{1}{\sqrt{9}} \\
&= \frac{1}{3}.
\end{aligned}$$

5. (20 pts) Let $f(x) = x^2 + 3x + 5$.

(a) Find the derivative $f'(x)$ using the *definition*.

**Solution 1:**

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 + 3(x+h) + 5 - (x^2 + 3x + 5)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h - x^2 - 3x - 5}{h} \\
&= \lim_{h\to 0} (2x + h + 3) \\
&= 2x + 3.
\end{aligned}$$

**Solution 2:**

$$\begin{aligned}
f'(x) &= \lim_{z\to x} \frac{f(z) - f(x)}{z - x} \\
&= \lim_{z\to x} \frac{z^2 + 3z + 5 - (x^2 + 3x + 5)}{z - x} \\
&= \lim_{z\to x} \frac{z^2 - x^2 + 3z - 3x}{z - x} \\
&= \lim_{z\to x} \frac{(z+x)(z-x) + 3(z-x)}{z - x} \\
&= \lim_{z\to x} (z + x + 3) \\
&= 2x + 3.
\end{aligned}$$

(b) Find the equation of tangent line to $y = f(x)$ at $x = 1$.

**Solution:** $a = 1$, $f(1) = 1^2 + 3 \cdot 1 + 5 = 9$ and $f'(1) = 2 \cdot 1 + 3 = 5$, so from the slope-point form

$$y - f(a) = f'(a)(x - a),$$

we obtain

$$y - 9 = 5(x - 1)$$

or

$$y = 5x + 4.$$

6. (10 pts) [Optional Bonus Question] Prove that

$$\lim_{x\to 0} x^4 \cos\frac{2}{x} = 0.$$

**Solution:** Since $-1 \le \cos\frac{2}{x} \le 1$,

$$-x^4 \le x^4 \cos\frac{2}{x} \le x^4.$$

$\lim_{x\to 0}(-x^4) = \lim_{x\to 0} x^4 = 0$, so by Squeeze Theorem,

$$\lim_{x\to 0} x^4 \cos\frac{2}{x} = 0.$$

7. (10 pts) [Optional Bonus Question] Find the limit if it exists. If not, explain why.

$$\lim_{x\to 2} \frac{|x-2|}{x-2}$$

**Solution:**

$$\begin{aligned}
\lim_{x\to 2+} \frac{|x-2|}{x-2} &= \lim_{x\to 2+} \frac{x-2}{x-2} \\
&= 1, \\
\lim_{x\to 2-} \frac{|x-2|}{x-2} &= \lim_{x\to 2-} \frac{-(x-2)}{x-2} \\
&= -1.
\end{aligned}$$

Since the left-hand limit and the right-hand limit do not coincide, the limit does not exist.